Fractional nonlinear diffusion equation: numerical analysis and large time behavior

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Séminaire Equations aux dérivées partielles
06/05/2025

Outline

$$\partial_t(v^{q-1}) + (-\Delta)^{\frac{\alpha}{2}} v = 0.$$

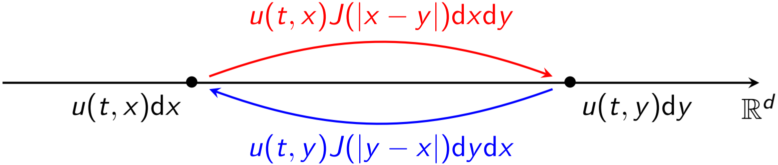
- $\alpha \in (0, 2)$
- $q \in (1, 2)$: fractional porous medium equation,
- $q \in (2, +\infty)$: fractional fast diffusion equation.

Purpose:

- Energy decay estimates,
- Numerical scheme preserving energy decay estimates,
- Large time behavior.

1. Background on fractional nonlinear diffusion equation
2. Numerical analysis of fractional nonlinear diffusion equation
3. Computation of extinction time and large time asymptotics

Nonlocal diffusion equation with general interaction kernel



$u(t,x)$: density, $J(|x-y|)$: interaction kernel.

$$\begin{aligned}\partial_t u(t,x) &= \int_{\mathbb{R}^d} u(t,y)J(|x-y|)\mathrm{d}y - u(t,x)\int_{\mathbb{R}^d} J(|x-y|)\mathrm{d}y \\ &= \text{P.V.}\int_{\mathbb{R}^d}(u(t,y)-u(t,x))J(|x-y|)\mathrm{d}y \\ &:= \lim_{r\to 0}\int_{\mathbb{R}^d\setminus B_r(x)}(u(t,y)-u(t,x))J(|x-y|)\mathrm{d}y\end{aligned}$$

for singular kernel J

A particular choice of kernel: the fractional diffusion equation

Choice of kernel: $J_{\frac{\alpha}{2}}(|x-y|) = \frac{1}{|x-y|^{d+\alpha}}$, $\quad \alpha \in (0,2)$, $\quad d$: dimension

Definition (Fractional Laplacian)

$$(-\Delta)^{\frac{\alpha}{2}} u(x) := C_{d,\alpha} \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}} \mathrm{d}y,$$

$$C_{d,\alpha} := \frac{\alpha 2^{\alpha-1} \Gamma\left(\frac{\alpha+d}{2}\right)}{\pi^{d/2} \Gamma\left(\frac{2-\alpha}{2}\right)}.$$

- $\mathcal{F}[(-\Delta)^{\frac{\alpha}{2}} u](\xi) = |\xi|^{\alpha} \mathcal{F}[u](\xi)$,
- $J_{\frac{\alpha}{2}}$ has a non-integrable singularity, and is heavy-tailed.

Fractional diffusion equation: $\partial_t u + (-\Delta)^{\frac{\alpha}{2}} u = 0$.

Nonlinear diffusion equation

$$\boxed{\partial_t u \quad - \quad \underbrace{\Delta(|u|^{m-1}u)}_{\nabla\cdot(D(u)\nabla u)} \; = 0}$$

$D(u) := \dfrac{|u|^{m-1}}{m}$: diffusion coefficient

$m > 1$	$0 < m < 1$
Porous medium equation	Fast diffusion equation
$\lim_{u\to 0} D(u) = 0$	$\lim_{u\to 0} D(u) = +\infty$
Finite speed of propagation...	Extinction phenomenon...

Shorthand: $u^m = |u|^{m-1}u \quad \rightarrow \quad \boxed{\partial_t u - \Delta u^m = 0}$

Fractional nonlinear diffusion equation on bounded domain

$$\partial_t u + (-\Delta)^{\frac{\alpha}{2}} u^m = 0$$

Change of variable: $q := 1/m + 1, \ v = u^m$

Fractional nonlinear diffusion equation on a bounded domain Ω

$$\begin{cases} \partial_t v^{q-1} + (-\Delta)^{\frac{\alpha}{2}} v = 0 & \text{in } \Omega \times (0, +\infty), \\ v = 0 & \text{in } (\mathbb{R}^d \setminus \Omega) \times (0, +\infty), \\ v(\cdot, 0) = v_0 & \text{in } \Omega. \end{cases} \tag{CDP}$$

- $q \in (1, 2)$: fractional porous medium equation,
- $q \in (2, +\infty)$: fractional fast diffusion equation.

Energy decay estimates: porous medium case

Proposition ([Bonforte, Vazquez], [Akagi, S.])

Assume $q \in (1,2)$*. Let* v *be an energy solution of* (CDP)*. There exist* $c, C > 0$ *such that, for any* $t > 0$*,*

$$\left(\|v^0\|_{L^q(\mathbb{R}^d)}^{q-2} + ct \right)^{\frac{1}{q-2}} \leq \|v(t)\|_{L^q(\mathbb{R}^d)} \leq \left(\|v^0\|_{L^q(\mathbb{R}^d)}^{q-2} + Ct \right)^{\frac{1}{q-2}} .$$

Energy decay estimates: fast diffusion case

Proposition ([Bonforte, Ibarrondo, Ispizua], [Akagi, S.])

Assume $q \in (2, 2_\alpha^*]$. *Let* v *be an energy solution of* (CDP). *There exist* $c, C > 0$ *such that, for any* $t > 0$,

$$\left(\|v_0\|_{L^q(\mathbb{R}^d)}^{q-2} - ct \right)_+^{\frac{1}{q-2}} \leq \|v(t)\|_{L^q(\mathbb{R}^d)} \leq \left(\|v_0\|_{L^q(\mathbb{R}^d)}^{q-2} - Ct \right)_+^{\frac{1}{q-2}}.$$

In particular, u *extincts at a time* $t_* \leq T_* := \frac{\|v^0\|_{L^q(\mathbb{R}^d)}^{q-2}}{C}$.

Moreover, for any $t > 0$,

$$c(t_* - t)_+^{\frac{1}{q-2}} \leq \|v(t)\|_{L^q(\mathbb{R}^d)} \leq C(t_* - t)_+^{\frac{1}{q-2}},$$

and the same is true when $\| \cdot \|_{L^q(\mathbb{R}^d)}$ is replaced by $\| \cdot \|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$.

Idea of proof

Lemma (Fractional Sobolev inequality)

Let $2^*_\alpha := \frac{2d}{(d-\alpha)_+}$, *and* $q \in (1, 2^*_\alpha]$. *There exists* $K > 0$ *such that,*

$$\|u\|_{L^q(\mathbb{R}^d)} \leq K[u]_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} = K\sqrt{\left((-\Delta)^{\frac{\alpha}{2}} u, u\right)_{L^2(\mathbb{R}^d)}},$$

for any $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ *with* $u \equiv 0$ *in* $\mathbb{R}^d \setminus \Omega$.

1. Obtain energy identity from the variational form of the equation,
2. Use the fractional Sobolev inequality, or monotonicity of Rayleigh quotient, to obtain an ordinary differential inequality,
3. Integrate the ordinary differential inequality.

Energy inequalities

Proposition ([Akagi, S.])

Let $v_0 \in H^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ *such that* $u_0 \equiv 0$ *in* $(\mathbb{R}^d \setminus \Omega) \times (0, +\infty)$ *and*

$$|v_0|^{q-2} v_0 \in L^{(2^*_\alpha)'}(\Omega) \quad \text{if } q > 2^*_\alpha.$$

There exists a unique weak solution to (CDP). *It satisfies*

$$\frac{1}{q'}\|v(t)\|_q^q + \int_s^t [v(r)]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \mathrm{d}r = \frac{1}{q'}\|v(s)\|_q^q \quad \text{for any } s \le t$$

$$\frac{4}{qq'}\int_s^t \|\partial_t(v^{\frac{q}{2}})(r)\|_2^2 \mathrm{d}r + \frac{1}{2}[v(t)]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \le \frac{1}{2}[v(s)]_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \quad \text{for any } s \le t$$

Idea of proof of existence and *inequalities*: Pass to the limit in the minimizing movement scheme

$$\frac{(v_{n+1})^{q-1} - (v_n)^{q-1}}{\Delta t} + (-\Delta)^{\frac{\alpha}{2}} v^{n+1} = 0.$$

Decay of Rayleigh quotient

Proposition

Assume $q \in (1, \infty)$. Let v be an energy solution to (CDP). *Define the Rayleigh quotient by*

$$R(t) = \frac{[v(t)]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}}{\|v(t)\|^2_{L^q(\mathbb{R}^d)}}, \quad t \geq 0.$$

Then $t \in [0, \infty) \mapsto R(t)$ is non-increasing as far as $u(t) \not\equiv 0$.

Idea of proof:

1. Show $\frac{\mathrm{d}R(t)}{\mathrm{d}t} \leq 0$ a.e. using energy inequalities,
2. Use absolute continuity of $t \mapsto \|v(t)\|_q$, decay of $t \mapsto [v(t)]_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$ and right-continuity of $t \mapsto R(t)$.

Outline of proof

$$R(t) = \frac{f(t)}{g(t)},$$

$$f(t) := [v(t)]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \in BV(0,T), \quad g(t) := \|v(t)\|_q^2 \in AC(0,T).$$

1. Show $\frac{\mathrm{d}R(t)}{\mathrm{d}t} \leq 0$ a.e. using energy inequalities,
2. Lebesgue decomposition's theorem:

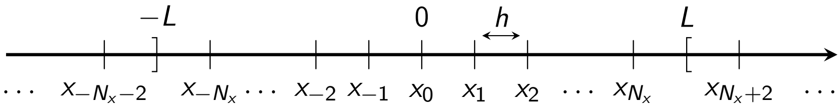
$$\mathrm{D}R = g\frac{\mathrm{d}f}{\mathrm{d}t}\mathcal{L}^1 + g(\mathrm{D}f)_s + f\frac{\mathrm{d}g}{\mathrm{d}t}\mathcal{L}^1 = \frac{\mathrm{d}R}{\mathrm{d}t} + g(\mathrm{D}f)_s,$$

3. Decay of $t \mapsto [v(t)]_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$: $(\mathrm{D}f)_s \leq 0$
4. R right-continuous: $R(t) - R(s) = \mathrm{D}R((s,t]) \leq 0$.

Notation

We restrict to dimension $d = 1$.

- $\Omega = (-L, L)$, space step $h = L/(N_x + 1)$:



- time step $\Delta t > 0$.

For u a fonction over $[0, +\infty) \times \mathbb{R}$,

$$u_i^n := u(n\Delta t, ih),$$

$$\|u^n\|_{l_h^q(\mathbb{R})}^q := \sum_{i \in \mathbb{Z}} |u_i|^q h.$$

Discretization of the fractional Laplacian

$$(-\Delta)^{\frac{\alpha}{2}} u(x_i) := C_{d,\alpha} \Bigg(\underbrace{\text{P.V.} \int_{|x_i-y|<h} \frac{u_i - u(y)}{|x_i - y|^{d+\alpha}} \mathrm{d}y}_{\text{singular part}} + \underbrace{\int_{|x_i-y|>h} \frac{u_i - u(y)}{|x_i - y|^{d+\alpha}} \mathrm{d}y}_{\text{tail part}} \Bigg)$$

- singular part: $u(y)$ replaced by Taylor expansion,
- tail part: $u(y)$ replaced by piecewise quadratic interpolation.

Then integrating explicitly yields, for some weights $(\gamma_j^h)_{j\in\mathbb{Z}}$,

$$(-\Delta)^{\frac{\alpha}{2}} u(x_i) \approx \sum_{j\in\mathbb{Z}} \gamma_j^h (u_i - u_{i-j}).$$

Convergence result: For $u \in C^4$, the error is in $\mathcal{O}(h^{3-\alpha})$.

➢ Y. Huang and A. Oberman. "Numerical Methods for the Fractional Laplacian: A Finite Difference-Quadrature Approach", 2014.

Convolution structure of the discrete fractional Laplacian

$$\left[(-\Delta)_h^{\frac{\alpha}{2}} u\right]_i := \sum_{j\in\mathbb{Z}} \gamma_j^h (u_i - u_{i-j})$$

$$\updownarrow$$

$$(-\Delta)^{\frac{\alpha}{2}} u(x_i) = \text{P.V.} \int_{\mathbb{R}^d} \frac{C_{1,\alpha}}{|z|^{1+\alpha}} (u(x_i) - u(x_i - z)) \mathrm{d}z.$$

Theorem ([Ayi, Herda, Hivert, Tristani, 2022])

There exists positive constants b_α and B_α independent of h such that

$$\frac{b_\alpha}{|jh|^{1+\alpha}} h \le \gamma_j^h \le \frac{B_\alpha}{|jh|^{1+\alpha}} h.$$

Convolution structure of the discrete fractional Laplacian

For u a Schwartz function,

$$\sum_{i\in\mathbb{Z}} h\left[(-\Delta)_h^{\frac{\alpha}{2}} u\right]_i u_i = \underbrace{\frac{1}{2}\sum_{i\in\mathbb{Z},j\in\mathbb{Z}} h\gamma_j^h |u_i - u_{i-j}|^2}_{=:[u]^2_{H_h^{\frac{\alpha}{2}}(\mathbb{R})}} \sim [u]^2_{H^{\frac{\alpha}{2}}(\mathbb{R})},$$

Lemma ([Ciaurri, Roncal, Stinga, Torrea, Varona] Discrete fractional Sobolev inequality)

For $q \leq 2^*_\alpha$, there exists $K > 0$ independent of h,

$$\|u\|_{l_h^q(\mathbb{R})} \leq K[u]_{H_h^{\frac{\alpha}{2}}(\mathbb{R})},$$

for $u \in \mathbb{Z}^{\mathbb{N}}$ with $u \equiv 0$ outside Ω.

Numerical scheme for fractional nonlinear diffusion equation

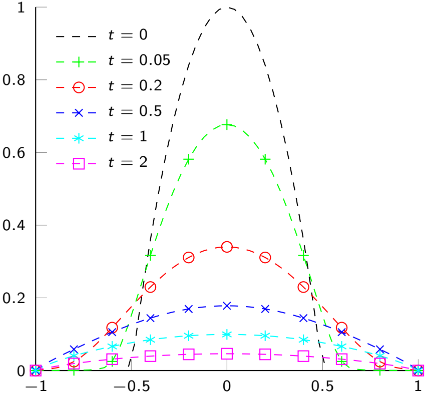
Implicit scheme for CDP

$$
\begin{aligned}
\frac{(u_i^{n+1})^{q-1}-(u_i^n)^{q-1}}{\Delta t}+\left[(-\Delta)_h^{\frac{\alpha}{2}}u^{n+1}\right]_i &= 0, && |i|\le N_x \text{ and } n\ge 0,\\
u_i^n &= 0, && |i|\ge N_x+1 \text{ and } n\ge 0,\\
u_i^0 &= (u^0)_i && |i|\le N_x.
\end{aligned}
$$

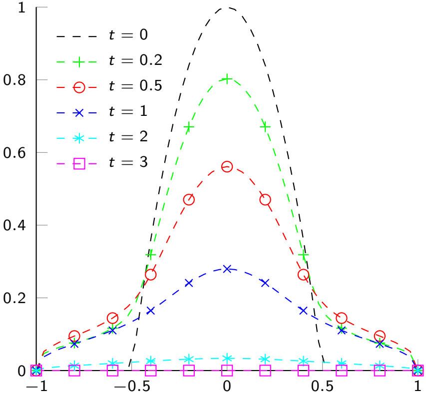
Properties as the continuous equation:

- decay estimates,
- almost extinction phenomenon in FDE case.

Figure: $\alpha = 0.5, h = 0.04, \Delta t = 0.001, L = 1$



(a) $q = 1.5$



(b) $q = 2.4$

Discrete decay estimates: porous medium case

Proposition ([Hivert, S.])

Assume $q \in (1,2)$. There exists $(\beta_n^{\Delta t})_{n\geq 0}$, independent of h, such that

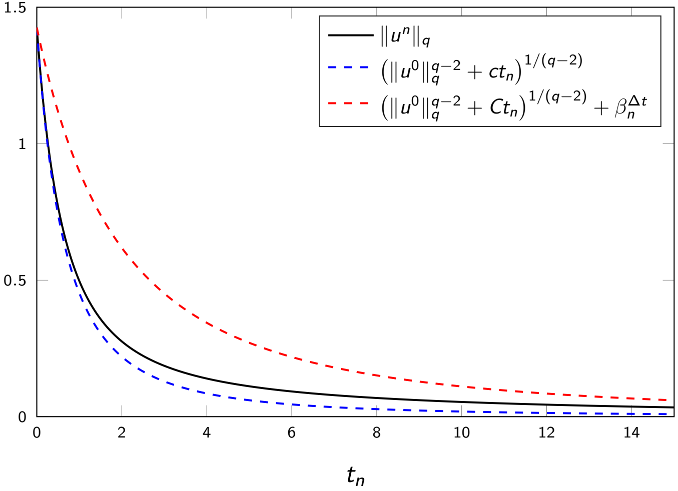
$$\|u^n\|_{l_h^q(\mathbb{R})} \leq \left(\|u^0\|_{l_h^q(\mathbb{R})}^{q-2} + Cn\Delta t\right)^{\frac{1}{q-2}} + \beta_n^{\Delta t}, \quad \textit{for any } n \geq 0.$$

Moreover,

$$\sup_{n\geq 0} \beta_n^{\Delta t} \to 0, \quad \textit{as } \Delta t \to 0.$$

Numerical results: energy decay for PME

Figure: Energy decay for $q = 1.5, \alpha = 0.5, h = 0.04, \Delta t = 0.03,\ L = 5.$



Discrete decay estimates: fast diffusion case

Proposition ([Hivert, S.])

Assume $q \in (2, 2^*_\alpha]$.

- *Decay estimate: There exists* $(\beta_n^{\Delta t})_{n\geq 0}$*, independent of* h*, such that*

$$\|u^n\|_{l_h^q(\mathbb{R})} \leq \left(\|u^0\|_{l_h^q(\mathbb{R})}^{q-2} - Cn\Delta t\right)_+^{\frac{1}{q-2}} + \beta_n^{\Delta t}, \quad \textit{for any } n \geq 0.$$

  Moreover,

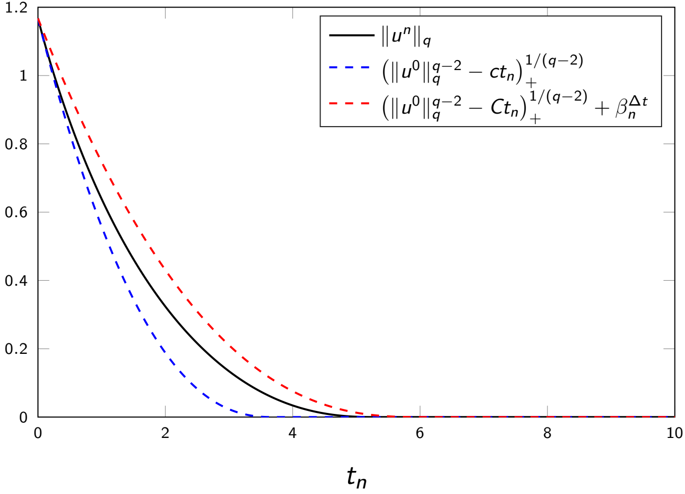
$$\sup_{n\geq 0} \beta_n^{\Delta t} \to 0, \quad \textit{as } \Delta t \to 0.$$

- *Extinction estimate:*

$$\|u^n\|_{l_h^q(\mathbb{R})} \leq \|u^0\|_{l_h^q(\mathbb{R})} \left(\frac{T_*}{n\Delta t}\right)^{n/2}, \quad \textit{for any } n \geq 0.$$

Numerical results: energy decay for FDE

Figure: Energy decay for $q = 2.4, \alpha = 0.5, h = 0.04, \Delta t = 0.03,\ L = 5.$



Idea of proof

1. Obtain energy inequality,

$$\frac{1}{q'}\frac{\|u^{n+1}\|^q_{l^q_h(\mathbb{R})} - \|u^n\|^q_{l^q_h(\mathbb{R})}}{\Delta t} + \|u^{n+1}\|^2_{H_h^{\frac{\alpha}{2}}(\mathbb{R})} \leq 0.$$

2. Use discrete fractional Sobolev inequality to obtain a discretization of the ordinary differential inequality,

$$\frac{1}{q'}\frac{\|u^{n+1}\|^q_{l^q_h(\mathbb{R})} - \|u^n\|^q_{l^q_h(\mathbb{R})}}{\Delta t} + K^{-2}\|u^{n+1}\|^2_{l^q_h(\mathbb{R})} \leq 0.$$

3. Sum in time and use convexity inequalities.

Convergence results

Theorem ([Hivert, S.] Convergence in time)

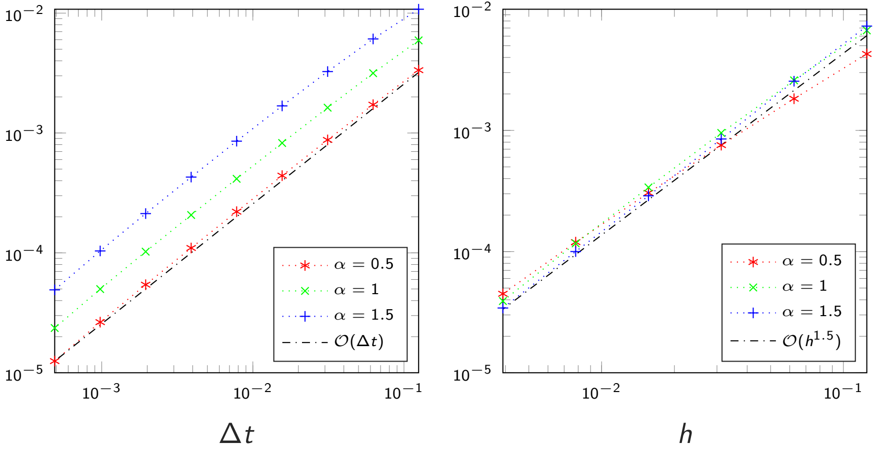
Let $u_i^{\Delta t}(t) = u_i^{n+1}$ *when* $t \in (n\Delta t, (n+1)\Delta t]$. *Then* $u_i^{\Delta t} \to u_i$ *uniformly on* $[0, T]$ *for any* $T > 0$, *where* $(u_i)_{i\in\mathbb{Z}}$ *is the solution of the semi-discrete scheme*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u_i^{q-1}(t) + \left[(-\Delta)_h^{\frac{\alpha}{2}} u(t)\right]_i = 0, & \text{for } |i| \leq N_x, \text{ and } t > 0, \\ u_i(t) = 0 & \text{for } |i| \geq N_x + 1, \text{ and } t \geq 0, \\ u_i(0) = (u_0)_i & \text{for } |i| \leq N_x. \end{cases}$$

Theorem ([Hivert, S.] Lax-Wendroff type theorem)

Let $u_{\Delta t,h}(x,t) = u(t_n, x_i)$ *for* $(t,x) \in [t_n, t_{n+1}) \times [x_i - h/2, x_i + h/2)$. *Assume* $\|u_{\Delta t,h}\|_\infty < +\infty$, *and* $u_{\Delta t,h} \to u$ *almost everywhere when* $(\Delta t, h) \to 0$. *Then* u *is a weak solution of* (CDP).

Figure: Convergence of the scheme for the norm $\| \cdot \|_{l^{\infty}_{\Delta t}(0,2,l^q_h(\mathbb{R}))}$
$q = 1.5, L = 1$



(a) $h = 0.05$, $\Delta t_{\mathrm{ref}} = 5 \cdot 10^{-5}$ (b) $\Delta t = 0.01$, $h_{\mathrm{ref}} = 2^{-11}$

Rescaled problem

Decay Estimates:

$$2 < q \le 2_\alpha^* : \quad c(t_* - t)_+^{\frac{1}{q-2}} \le \|v(t)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \le C(t_* - t)_+^{\frac{1}{q-2}}.$$

$$1 < q < 2 : \quad c(1+t)^{\frac{1}{q-2}} \le \|v(t)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \le C(1+t)^{\frac{1}{q-2}}.$$

Rescaled solution:

$$2 < q < 2_\alpha^* : \quad w(s) := (t_* - t)^{\frac{-1}{q-2}} v(t), \quad s := \log\left(\frac{t_*}{t_* - t}\right).$$

$$1 < q < 2 : \quad w(s) := (1+t)^{\frac{-1}{q-2}} v(t), \quad s := \log(t+1).$$

Then $\partial_s w^{q-1} + (-\Delta)^{\frac{\alpha}{2}} w = \dfrac{q-1}{|q-2|} w^{q-1}$, and $c < \|w(s)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < C$

Does $w(s)$ converge as $s \to \infty$?

Rescaled problem

$$\partial_s w^{q-1} + (-\Delta)^{\frac{\alpha}{2}} w = \frac{q-1}{|q-2|} w^{q-1}, \text{ and } c < \|w(s)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < C$$

Non-fractional FDE case:

- Convergence along subsequences: ➢ Berryman and Holland 1980
- Convergence along the full sequence: ➢ Feireisl and Simondon 2000
- Convergence in relative error: ➢ Bonforte, Grillo, and Vazquez 2012
- Sharp rate of convergence: ➢ Bonforte and Figalli 2021
  ➢ Jin and Xiong 2023
  ➢ Akagi 2023
  ➢ Choi, McCann, and Seis 2023

Generalized gradient flow structure for nonlinear diffusion

Define

$$J(w) := \frac{1}{2}[w]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} - \frac{q-1}{q|q-2|}\|w\|^q_{L^q(\mathbb{R}^d)}.$$

Then the rescaled solution $s > 0 \mapsto w(s)$ solves

$$\partial_s w^{q-1}(s) = -J'(w(s)), \quad \text{for a.e. } s > 0.$$

Therefore it holds

$$\frac{4}{qq'}\left\|\partial_s(|w|^{(q-2)/2}w)(s)\right\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}s}J(w(s)) \leq 0, \quad \text{for a.e. } s > 0,$$

and $J(w(\cdot))$ is non-increasing.

Convergence along subsequences

$$\partial_s w^{q-1} + (-\Delta)^{\frac{\alpha}{2}} w = \frac{q-1}{|q-2|} w^{q-1}, \text{ and } c < \|w(s)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < C.$$

Proposition ([Akagi, S.])

For any $s_n \to +\infty$, there exists a subsequence (still denoted by (s_n)), and $\phi \in H^{\frac{\alpha}{2}}(\mathbb{R}^d) \setminus \{0\}$, with $\phi = 0$ in $\mathbb{R}^d \setminus \Omega$, such that

$$w(s_n) \to \phi \quad \text{strongly in } H^{\frac{\alpha}{2}}(\mathbb{R}^d),$$
$$(-\Delta)^{\frac{\alpha}{2}} \phi = \lambda_q \phi^{q-1} \quad \text{in } \Omega,$$

with $\lambda_q := \frac{q-1}{|q-2|} > 0$.

Full convergence:

- PME case: uniqueness of positive solution to the stationary problem.
- FDE case: Łojasiewicz-Simon inequality.

PME case: uniqueness to the stationary equation

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = f(x,u) & \text{in } \Omega, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases} \tag{E}$$

(F1) For a.e. $x \in \Omega$, $u \mapsto f(x,u)$ is continuous on $[0,\infty)$ and $u \mapsto f(x,u)/u$ is strictly decreasing on $(0,\infty)$.

(F2) $\forall u \in [0,\infty)$, $x \mapsto f(x,u)$ is in $L^\infty(\Omega)$.

(F3) $\exists C \in \mathbb{R}$ s.t. $\forall u \in [0,\infty)$, for a.e. $x \in \Omega$, $f(x,u) \leq C(|u|+1)$.

Proposition ([Brezis, Oswald], [Akagi,S.])

Let Ω is a bounded $C^{1,1}$ domain of $\mathbb{R}^d$ and assume (F1)–(F3). Then the problem (E) admits at most one weak solution $u \in H^{\alpha/2}(\mathbb{R}^d) \cap L^\infty(\Omega)$.

Outline of proof

Assume u, v are two solutions of (E).

1. Dividing the equation for u (resp. v) by u (resp. v) and substracting:

$$\frac{(-\Delta)^{\frac{\alpha}{2}} u(x)}{u(x)} - \frac{(-\Delta)^{\frac{\alpha}{2}} v(x)}{v(x)} = \frac{f(x, u(x))}{u(x)} - \frac{f(x, v(x))}{v(x)}$$

2. Strict sublinearity of f:

$$\int_\Omega \left(\frac{f(x, u(x))}{u(x)} - \frac{f(x, v(x))}{v(x)} \right) (u^2(x) - v^2(x)) \mathrm{dx} \leq 0,$$

   with equality iif $u = v$ a.e.

3. "Monotony" of $(-\Delta)^{\frac{\alpha}{2}}$:

$$\left\langle \frac{(-\Delta)^{\frac{\alpha}{2}} u}{u} - \frac{(-\Delta)^{\frac{\alpha}{2}} v}{v}, u^2 - v^2 \right\rangle_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \geq 0.$$

FDE case: Łojasiewicz-Simon inequality for fractional Laplacian

$$I(w) := \frac{1}{2}[w]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \int_\Omega \int_0^{w(x)} g(s)\mathrm{d}s\mathrm{d}x.$$

(H0) $g \in C^1(\mathbb{R})$ and $g(0) = 0$,
(H1) $g \in C^\infty((0,\infty))$, and for all $\beta \in (0,\infty)$, there exist $C, M \geq 0$ such that,

$$|g^{(n)}(s)| \leq C\frac{M^n n!}{|s|^n}, \quad \forall s \in (0,\beta),\ n \in \mathbb{N}$$

(H2) there exists $0 \leq p < \infty$ with $p \leq 2^*_\alpha - 1$ such that

$$|g'(s)| \leq C(|s|^{p-1} + 1) \quad \text{for all } s \in \mathbb{R}.$$

Lemma ([Akagi, Schimperna, Segatti])

Assume (H0), (H1), (H2), and let $\psi \in H^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^\infty(\Omega)$ such that $I'(\psi) = 0$ and $\psi > 0$. Then, there exists $\theta \in (0, 1/2]$ and $\omega, \delta > 0$ s.t.

$$|I(w) - I(\psi)|^{1-\theta} \leq \omega \|I'(w)\|_{H^{-\frac{\alpha}{2}}(\Omega)}, \quad \text{if } \|w - \psi\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < \delta.$$

FDE case: Łojasiewicz inequality

Theorem ([Łojasiewicz])

Let $U \subset \mathbb{R}^d$ open and $f : U \to \mathbb{R}$ a real-analytic function. Let $x_0 \in U$ such that $\nabla f(x_0) = 0$. Then there exists a neighborhood V of x_0, $\omega > 0$ and $\theta \in (0, 1/2]$ such that

$$|f(x) - f(x_0)|^{1-\theta} \leq \omega |\nabla f(x)|, \quad x \in V.$$

Proof for $d = 1$: $\exists N \geq 2$ s.t. $f^{(N)}(x_0) \neq 0$ and

$$f(x_0 + h) - f(x_0) = \frac{f^{(N)}(x_0)}{N!} h^N + o(h^N),$$

$$f'(x_0 + h) = \frac{f^{(N)}(x_0)}{(N-1)!} h^{N-1} + o(h^{N-1})$$

$$\Rightarrow (f(x_0 + h) - f(x_0))^{\frac{N-1}{N}} = \frac{N!^{1/N}}{N f^{(N)}(x_0)^{1/N}} f'(x_0 + h) + \underbrace{o(h^{N-1})}_{\lesssim f'(x_0+h)}$$

FDE case: Idea of proof of full convergence

Assume $v(s_n) \to \phi \in H^{\frac{\alpha}{2}}(\mathbb{R}^d)$, with $\phi > 0$ and $(-\Delta)^{\frac{\alpha}{2}}\phi = \frac{q-1}{q-2}\phi^{q-1}$ in Ω.

Energy inequality and Łojasiewicz-Simon inequality:

$$\frac{4}{qq'}\|\partial_s w^{q/2}(s)\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}s}J(w(s)) \le 0, \tag{1}$$

$$|J(w) - J(\phi)|^{1-\theta} \le \omega \|J'(w)\|_{H^{-\frac{\alpha}{2}}(\Omega)}, \quad \text{if } \|w-\phi\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < \delta \tag{2}$$

1. Using (1),(2), chain rule and Poincaré-Sobolev inequality, $\exists c_0 > 0$ s.t.

$$\|w(s)-\phi\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < \delta \Rightarrow c_0\|\partial_s(w^{q-1})(s)\|_{H^{-\frac{\alpha}{2}}(\Omega)} \le -\frac{\mathrm{d}}{\mathrm{d}s}\left(J(w(s)) - J(\phi)\right)^{\theta}.$$

2. If $\exists s_0$ s.t. $\forall s > s_0$ $\|w(s)-\phi\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < \delta$, then

$$c_0\int_s^\infty \|\partial_s(w^{q-1})(s)\|_{H^{-\frac{\alpha}{2}}(\Omega)}\mathrm{d}s \le \left(J(w(s)) - J(\phi)\right)^{\theta} \to 0 \text{ as } s \to \infty.$$

3. If not, we obtain a contradiction extracting a subsequence $\tilde{s_n} \in (s_n, s_{n+1})$ such that $\|w(\tilde{s_n})-\phi\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} = \delta/2$

Full convergence to the asymptotic profile

Proposition ([Akagi, S.])

Let $q \in (1, 2_\alpha^*) \setminus \{2\}$. Assume that $\phi > 0$ and $w(s_n) \to \phi$ strongly in $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ for some sequence of times $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to +\infty$. Then

$$w(s) \to \phi \quad \textit{strongly in } H^{\frac{\alpha}{2}}(\mathbb{R}^d) \textit{ as } s \to +\infty.$$

Idea of proof. $1 < q < 2$: uniqueness to the stationary equation.
$2 < q < 2_\alpha^*$: Łojasiewicz-Simon inequality with the functionnal

$$J(w) := \frac{1}{2}[w]^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} - \frac{\lambda_q}{q}\|w\|^q_{L^q(\mathbb{R}^d)}.$$

Open problems:

- Computation of the extinction time,
- Rate of convergence to the asymptotic profile in FDE case.

Non-fractional FDE: optimal rate of convergence

Assume $q \in (2, 2^*)$.

Linearized equation:

$$\begin{cases} (q-1)\phi^{q-2}\partial_s h = -\mathcal{L}_{\phi,2} h & \text{in } \Omega \times (0, +\infty), \\ h = 0 & \text{in } \partial\Omega \times (0, +\infty), \end{cases}$$

with $\mathcal{L}_{\phi,\alpha} h = (-\Delta)^{\alpha/2} h - (q-1)\lambda_q \phi^{q-2} h$, and $h(s) \approx w(s) - \phi$.

Theorem ([Bonforte, Figalli],[Akagi])

Assume $\alpha = 2$ and 0 is not an eigenvalue of $\mathcal{L}_{\phi,2}$, and let ν_k be the first positive eigenvalue of $\mathcal{L}_{\phi,2}/((q-1)\phi^{q-2})$. Then, there exists a constant $C > 0$ such that

$$\left(\int_\Omega |\nabla w(x,s) - \nabla \phi|^2 \mathrm{d}x \right)^{1/2} \le C e^{-\nu_k s}, \quad \text{for } s \ge 0.$$

Numerical approximation of extinction time when $q > 2$

v : solution to continuous problem with initla data v^0
$t_*(v^0)$: extinction time of v
$\tilde{t}$: approximation of $t_*(v^0)$

Rescaling with $\tilde{t}$: $w(s) := (\tilde{t} - t)^{\frac{-1}{q-2}} v(t)$, $s := \log(\tilde{t}/(\tilde{t} - t))$.

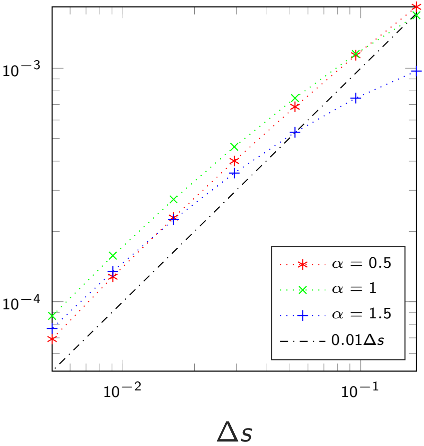
- $\partial_s w^{q-1} + (-\Delta)^{\frac{\alpha}{2}} w = \frac{q-1}{q-2} w^{q-1}, \quad w(0) = \tilde{t}^{\frac{-1}{q-2}} v^0$
- $\tilde{t} < t_*(v^0) \Rightarrow \|w(s)\|_q \to \infty$ as $s \to \infty$,
- $\tilde{t} > t_*(v^0) \Rightarrow w$ extincts in finite time.

Computation of $t_*(v^0)$: dichotomy using the scheme

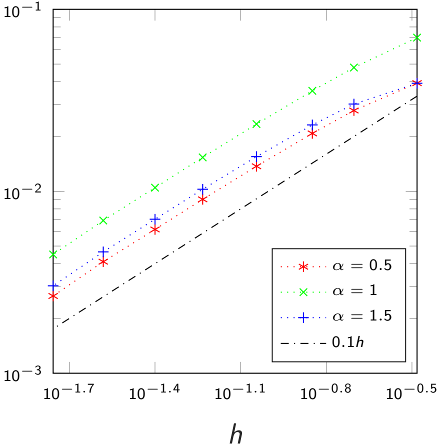
$$\begin{cases} \dfrac{(w_i^{n+1})^{q-1} - (w_i^n)^{q-1}}{\Delta s} + \left[(-\Delta)_h^{\frac{\alpha}{2}} w^{n+1}\right]_i = \dfrac{q-1}{q-2}(w^{n+1})^{q-1}, \\ \hfill w_i^0 = \tilde{t}^{-1/(q-2)}(v^0)_i. \end{cases}$$

Figure: Convergence of extinction time computed by dichotomy

$q = 2.4, L = 1$

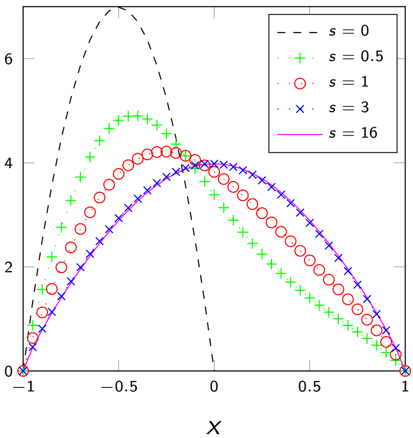


(a) Convergence in Δs for $h = 0.1$

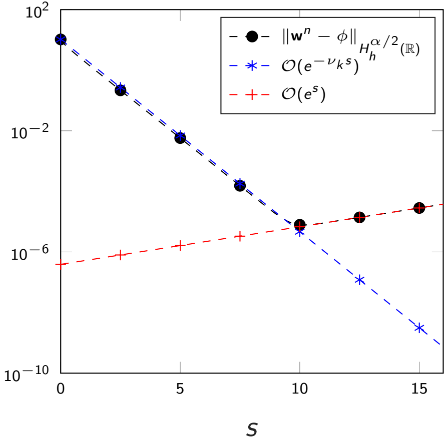


(b) Convergence in h for $\Delta s = 0.1$

Figure: Solution of rescaled scheme with t_* computed by dichotomy
$\alpha = 1.5$, $q = 2.4 < 2^*_\alpha$, $h = 0.01$, $\Delta s = 0.01$



(a) Rescaled solution $\mathbf{w}^n$



(b) Error with asymptotic profile ϕ
ν_k : first positive eigenvalue of the linearized problem

Conclusion

Summary:

- Decay estimates,
- Numerical scheme with same decay estimates,
- Convergence to asymptotic profile,
- Numerical method for computing extinction time in FDE case.

Extensions:

- Rates of convergence to asymptotic profiles in the fractional case,
- Numerical analysis in dimension $d \geq 1$ and better convergence results.

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Extensions:

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- Numerical analysis in dimension $d \geq 1$ and better convergence results.

Thank you for your attention!